

# Periodic Asynchronous Event-triggered Control <sup>★</sup>

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## Abstract

Asynchronous event-triggered control (AETC) is an implementation of controllers characterized by decentralized event generation, asynchronous sampling updates, and dynamic quantization. Combining those elements in AETC results in a parsimonious transmission of information which makes it suitable for wireless networked implementations. We extend the previous work on AETC by introducing periodic sampling, denoting our proposal periodic asynchronous event-triggered control (PAETC), and study the stability and  $\mathcal{L}_2$ -gain of PAETC for implementations affected by disturbances. In PAETC, at each sampling time, quantized measurements from those sensors that triggered a local event are transmitted to a dynamic controller that computes control actions; the quantized control actions are then transmitted to the corresponding actuators only if certain events are also triggered for the corresponding actuator. The developed theory is demonstrated and illustrated via a numerical example.

*Key words:* Decentralized event-triggered control; periodic sampling; dynamic quantization; wireless networked control systems; cyber-physical systems.

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## 1 Introduction

In digital control applications, the control task consists of sampling and transmitting the output of the plant, computing and implementing controller outputs. Current developments of sensor and networking technologies have enabled the emergence of wireless networked control systems (WNCS), in which communication of distributed components is established via wireless networks. WNCS can be established and updated with large flexibility and low cost, and are especially suitable to physically distributed plants. Limited energy supplies are often the case when sensors are battery powered for mobility and/or flexibility reasons. The major challenge in WNCS design is thus to achieve prescribed performance under limited bandwidth and limited energy supplies.

To reduce the bandwidth occupation a type of aperiodic control task execution strategy, named event-triggered control (ETC), has been proposed, see [4], [5], [19], [20], [23], [25], [26], and references there in. However, most of the existing event-triggered mechanisms require sensors to continuously sample the plant output to validate event conditions. This continuous sampling

results in large amounts of energy consumption [13]. To reduce sampling time in ETC, two solutions can be found in the literature: determine the sampling time a-priori by modeling the event times dynamics, see [15]; or apply periodic sampling (while transmission may remain aperiodic), see [11]. In [15], Kolarijani and Mazo study the inter-event times dynamics for event-triggered mechanisms like the ones from [23]. They approximate the inter-event times dynamics by constructing an approximate power quotient system, and determine the sampling time intervals on conic regions. Their work is extended to a class of perturbed systems in [16]. However, their approach can only be applied to state feedback systems, and the considered perturbed systems are limited to disturbances that vanish as the state converges to the origin. In [11], Heemels et. al. present a periodic event-triggered control (PETC) mechanism. In PETC, the sensors sample the output of the plant and verify the central or local event conditions periodically. Therefore, the energy consumed by sensing is reduced compared to those continuously monitoring event-triggered mechanisms, while still a pre-designed performance can be guaranteed. The disturbances considered in their work only require to have finite  $\mathcal{L}_2$  norm. Furthermore, the transmissions of the controller actions to actuators are also governed by an event-triggering mechanism. The main limitation of that work, which we address in the current work, is that it does not consider quantization: it assumes that updates occur instantaneously and accurately.

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Two main type of quantizers can be considered: static, see e.g. [3], [6], [8]; and dynamic, see e.g. [17], [18], [21]. In [18], Liberzon and Nešić present a state dependent quantizer which zooms in and out based on the system's state, so as to provide input to state stability (ISS). In [19], Mazo and Cao present an asynchronous event-triggered control (AETC) mechanism combining state dependent dynamic quantization and decentralized event-triggering conditions. In the AETC from [19], the event conditions are distributed to each sensor node employing a coordination mechanism. This coordination mechanism employs a global threshold which is updated depending on the current sampled state at the controller. Each sensor employs this global threshold to compute local event-triggering conditions. This approach allows each sensor to verify the triggering condition and transmit samples independently of each other. With such a shared global threshold, transmission of just one bit is necessary to update the controller. The transmission of this bit indicates that some error signal (the difference between the current and last measurement) has exceeded the value of a local-threshold and the value of that bit indicates the sign of that error signal. This is similar to the  $\Delta$ -sampling scheme e.g. [7]. Furthermore, the controller is only updated with the measurements from those sensors which triggered an event (thus the term asynchronous). This mechanism can largely reduce the required transmissions. However, [19] does not address neither systems with disturbances, nor systems with output maps different than the identity or dynamic controllers. Furthermore, this event-triggered mechanism also requires the sensors to monitor the plant outputs continuously to verify the event conditions, and the actuators to listen to the controller continuously to receive controller outputs. This AETC mechanism is extended in [12] which presents a scheme to monitor the plant outputs discontinuously. The approach taken there is to employ a smaller global threshold to compensate for the delay caused by the discontinuous samplings.

Our present work is mostly inspired by [11], [18], and [19]. We propose a periodic asynchronous event-triggered control (PAETC) mechanism building on the aforementioned pieces of work with the goal of reducing wireless channel bandwidth occupation and energy consumption. This PAETC incorporates: quantization in a zooming fashion, which is similar to [18] and [19]; an asynchronous event-triggered mechanism, based on [19]; and periodic sampling as in [11]. Moreover, compared with [18] and [19], in our approach the quantization error or global threshold depends on the information in the controller, instead of just on the current estimation of the system's state; compared with [11], in which the algorithm for designing decentralized event condition parameters is complex: requiring to solve a set of LMIs, our approach requires to solve only one LMI.

The organization of the remainder of the paper is as follows. The mathematical notation, problem setup, and

the necessary preliminary definitions and theorems are presented in Section II. The introduction of PAETC and problem formulation follows in Section III. Section IV presents the threshold update mechanism and performance function, and analyzes the system stability and  $\mathcal{L}_2$ -gain. The maximum packet size of each transmission is presented in Section V. Finally, an illustrative example is introduced in Section VI and the paper is concluded with a brief discussion in Section VII. The technical proofs of the lemmas and theorems are provided in an appendix at the end of the paper.

## 2 Notation and preliminaries

We denote the positive real numbers by  $\mathbb{R}^+$ , by  $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$ , and the natural numbers including zero by  $\mathbb{N}$ .  $|\cdot|$  denotes the Euclidean norm in the appropriate vector space, when applied to a matrix  $|\cdot|$  denotes the  $l_2$  induced matrix norm, when applied to a set  $|\cdot|$  denotes the cardinality of the set. A symmetric matrix  $P \in \mathbb{R}^{n \times n}$  is said to be positive definite, denoted by  $P \succ 0$ , whenever  $x^T P x > 0$  for all  $x \neq 0, x \in \mathbb{R}^n$ . A function  $\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  belongs to class  $\mathcal{K}(\alpha \in \mathcal{K})$  if:  $\alpha$  is a continuous function,  $\alpha(0) = 0$  and  $s_1 > s_2 \Rightarrow \alpha(s_1) > \alpha(s_2)$ . A function  $\alpha: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  belongs to class  $\mathcal{K}_\infty(\alpha \in \mathcal{K}_\infty)$  if:  $\alpha \in \mathcal{K}$  and  $\lim_{s \rightarrow \infty} \alpha(s) = \infty$ . A function  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  is positive definite ( $\varphi \in \mathcal{PD}$ ), if:  $\varphi(s) > 0$  for all  $s > 0$  and  $\varphi(0) = 0$ . For a locally integrable signal  $w: \mathbb{R}^+ \rightarrow \mathbb{R}^n$ , we denote by  $\|w\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty |w(t)|^2 dt}$  its  $\mathcal{L}_2$ -norm,  $\|w\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|w(t)\| < \infty$  its  $\mathcal{L}_\infty$ -norm. Furthermore, we define the space of all locally integrable signals with a finite  $\mathcal{L}_2$ -norm as  $\mathcal{L}_2$ .  $|x|_{\mathcal{A}} = \min\{|x - y| : y \in \mathcal{A}\}$  indicates the distance of the vector  $x$  to the closed set  $\mathcal{A}$ . For a symmetric matrix  $P$ ,  $\lambda_{\max}(P)$  and  $\lambda_{\min}(P)$  are the maximum and minimum eigenvalues of  $P$  respectively.  $\lfloor s \rfloor$  and  $\lceil s \rceil$  denote the nearest integer from below and above of a scalar  $s$  respectively.

Let us consider a linear time-invariant (LTI) plant given by:

$$\begin{cases} \dot{\xi}_p(t) = A_p \xi_p(t) + B_p \hat{v}(t) + E w(t) \\ y(t) = C_p \xi_p(t), \end{cases} \quad (1)$$

where  $\xi_p(t) \in \mathbb{R}^{n_p}$  and  $y(t) \in \mathbb{R}^{n_y}$  denote the state vector and output vector of the plant respectively,  $\hat{v}(t) \in \mathbb{R}^{n_v}$  denotes the input applied to the plant,  $w(t) \in \mathbb{R}^{n_w}$  denotes an unknown disturbance. The plant is controlled by a discrete-time controller given by:

$$\begin{cases} \xi_c(t_{k+1}) = A_c \xi_c(t_k) + B_c \hat{y}(t_k) \\ v(t_k) = C_c \xi_c(t_k) + D_c \hat{y}(t_k), \end{cases} \quad (2)$$

where  $\xi_c(t_k) \in \mathbb{R}^{n_c}$ ,  $v(t_k) \in \mathbb{R}^{n_v}$ , and  $\hat{y}(t_k) \in \mathbb{R}^{n_y}$  denote the state vector, output vector of the controller, and input applied to the controller respectively. Define  $h > 0$  the sampling interval. A periodic sampling sequence is

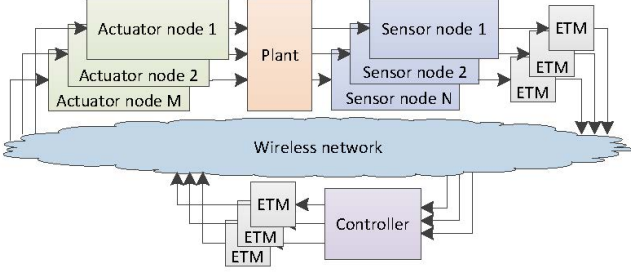


Figure 1. Wireless networked systems architecture, indicating the event-triggered mechanisms (ETMs)

given by:

$$\mathcal{T} := \{t_k | t_k := kh, k \in \mathbb{N}\}.$$

At each sampling time  $t_k$ , the input applied to the plant  $\hat{v}(t_k)$  and the input applied to the controller  $\hat{y}(t_k)$  are determined by:

$$\forall t_k \in \mathcal{T}, \hat{v}^i(t_k) := \begin{cases} q(v^i(t_k)), & \text{if there is an event} \\ \hat{v}^i(t_{k-1}), & \text{otherwise,} \end{cases} \quad (3)$$

$$\forall t_k \in \mathcal{T}, \hat{y}^i(t_k) := \begin{cases} q(y^i(t_k)), & \text{if there is an event} \\ \hat{y}^i(t_{k-1}), & \text{otherwise,} \end{cases} \quad (4)$$

where  $\hat{v}^i(t_k)$   $\hat{y}^i(t_k)$  denote the  $i$ -th entries of the vectors  $\hat{v}(t_k)$  and  $\hat{y}(t_k)$ , and  $q(s)$  denotes the quantized signal of  $s$ . Therefore, at each sampling time, only those inputs with events are required to transmit measurements or actuation signals through the network.

We also introduce a performance variable  $z \in \mathbb{R}^{n_z}$  given by:

$$z(t) = g(\xi(t), w(t)), \quad (5)$$

where  $\xi(t) := [\xi_p^T(t) \ \xi_c^T(t) \ \hat{y}^T(t) \ \hat{v}^T(t)]^T \in \mathbb{R}^{n_\xi}$ ,  $n_\xi := n_p + n_c + n_y + n_v$ ,  $g(s)$  is a design function.

The architecture of this implementation is shown in Fig. 1. In this implementation, the controller, sensors, and actuators are assumed to be physically distributed, and exchange information via a wireless network. Furthermore, we assume none of the nodes are co-located.

Some additional preliminaries are provided in the following: Denote  $\mathcal{H} := (C_H, F_H, D_H, G_H)$  a hybrid system [9], in which  $C_H$  is the flow set,  $F_H$  is the flow map,  $D_H$  is the jump set,  $G_H$  is the jump map. A subset  $E_H$  of  $\mathbb{R}_0^+ \times \mathbb{N}$  is a hybrid time domain [9], if it is the union of infinitely many intervals of the form  $[t_j, t_{j+1}] \times \{j\}$ , where  $0 = t_0 \leq t_1 \leq t_2 \leq \dots$ , or of finitely many such intervals, with the last one possible of the form  $[t_j, t_{j+1}] \times \{j\}$ ,  $[t_j, t_{j+1}) \times \{j\}$ , or  $[t_j, \infty) \times \{j\}$ . A hybrid arc [9] is a function  $\phi : \text{dom } \phi \rightarrow \mathbb{R}^n$ , where  $\text{dom } \phi$  is a hybrid time domain and, for each fixed  $j$ ,  $t \rightarrow \phi(t, j)$  is a locally absolutely continuous function on the interval

$I_j = \{t : (t, j) \in \text{dom } \phi\}$ . The hybrid arc  $\phi$  is a solution to the hybrid system  $\mathcal{H} = (C_H, F_H, D_H, G_H)$ , if  $\phi(0, 0) \in C_H \cup D_H$ .

**Definition 1** (Uniform global pre-asymptotic stability (UGpAS))[10] Consider a hybrid system  $\mathcal{H}$  on  $\mathbb{R}^n$ . Let  $\mathcal{A} \subset \mathbb{R}^n$  be closed. The set  $\mathcal{A}$  is said to be

- uniformly globally stable for  $\mathcal{H}$  if there exists a class- $\mathcal{K}_\infty$  function  $\alpha$  such that any solution  $\phi$  to  $\mathcal{H}$  satisfies  $|\phi(t, j)|_{\mathcal{A}} \leq \alpha(|\phi(0, 0)|_{\mathcal{A}})$  for all  $(t, j) \in \text{dom } \phi$ ;
- uniformly globally pre-attractive for  $\mathcal{H}$  if for each  $\varepsilon > 0$  and  $r > 0$  there exists  $T > 0$  such that, for any solution  $\phi$  to  $\mathcal{H}$  with  $|\phi(0, 0)|_{\mathcal{A}} \leq r$ ,  $(t, j) \in \text{dom } \phi$  and  $t+j \geq T$  imply  $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$ ;
- uniformly globally pre-asymptotically stable for  $\mathcal{H}$  if it is both uniformly globally stable and uniformly globally pre-attractive.

**Definition 2** (Lyapunov function candidate)[9] Given the hybrid system  $\mathcal{H} = (C_H, F_H, D_H, G_H)$ , and the compact set  $\mathcal{A} \subset \mathbb{R}^n$ , the function  $V : \text{dom } V \rightarrow \mathbb{R}$  is a Lyapunov function candidate for  $(\mathcal{H}, \mathcal{A})$  if the following conditions hold:  $V$  is continuous and nonnegative on  $(C_H \cup D_H) \setminus \mathcal{A} \subset \text{dom } V$ ;  $V$  is continuously differentiable on an open set  $\mathcal{O}$  satisfying  $C_H \setminus \mathcal{A} \subset \mathcal{O} \subset \text{dom } V$ ;  $\lim_{\{x \rightarrow \mathcal{A}, x \in \text{dom } V \cap (C_H \cup D_H)\}} V(x) = 0$ .

**Theorem 3** (Sufficient Lyapunov conditions)[9] Consider the hybrid system  $\mathcal{H} = (C_H, F_H, D_H, G_H)$  and the compact set  $\mathcal{A} \subset \mathbb{R}^n$  satisfying  $G_H(\mathcal{A} \cap D_H) \subset \mathcal{A}$ . If there exists a Lyapunov function candidate  $V$  for  $(\mathcal{H}, \mathcal{A})$  such that

$$\begin{aligned} \langle \nabla V(x), f \rangle &\leq 0, \forall x \in C_H \setminus \mathcal{A}, f \in F_H(x) \\ V(g) - V(x) &\leq 0, \forall x \in D_H \setminus \mathcal{A}, g \in G_H(x) \setminus \mathcal{A}. \end{aligned}$$

then  $\mathcal{A}$  is pre-asymptotically stable and the basin of pre-attraction contains every forward invariant, compact set.

**Definition 4** ( $\mathcal{L}_2$ -gain)[11] The system (1), (2), (5) is said to have an  $\mathcal{L}_2$ -gain from  $w$  to  $z$  smaller than or equal to  $\gamma$ , if there is a  $\mathcal{K}_\infty$  function  $\delta : \mathbb{R}^{n_\xi} \rightarrow \mathbb{R}^+$  such that for any  $w \in \mathcal{L}_2$ , any initial state  $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$  and  $\tau(0) \in [0, h]$ , the corresponding solution to system (1) (2) (5) satisfies  $\|z\|_{\mathcal{L}_2} \leq \delta(\xi_0) + \gamma\|w\|_{\mathcal{L}_2}$ .

### 3 Problem definition

Define two vectors for the implementation input and output:

$$\begin{aligned} u(t) &:= [y^T(t) \ v^T(t)]^T \in \mathbb{R}^{n_u} \\ \hat{u}(t_k) &:= [\hat{y}^T(t_k) \ \hat{v}^T(t_k)]^T \in \mathbb{R}^{n_u}, \end{aligned} \quad (6)$$

with  $n_u := n_y + n_v$ .  $u^i(t_k)$ ,  $\hat{u}^i(t_k)$  are the  $i$ -th elements of the vector  $u(t_k)$ ,  $\hat{u}(t_k)$  respectively. In the local event conditions in (3) and (4), an event is occurred when the following inequality holds:

$$|\hat{u}^i(t_{k-1}) - u^i(t_k)| \geq \sqrt{\eta_i(t_k)}, i \in \{1, \dots, n_u\}, \quad (7)$$

in which  $\eta_i(t_k)$  is a local threshold, computed as:

$$\eta_i(t) := \theta_i^2 \eta^2(t), \quad (8)$$

where  $\theta_i$  is a pre-designed distributed parameter satisfying  $|\theta| = 1$  and  $\eta : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ , determines the global threshold, which will be discussed in Section 4.

When an event takes place at a sampling time  $t_k$ ,  $\hat{u}(t_k)$  is updated by:

$$\begin{aligned} \hat{u}^i(t_k) &= q(u^i(t_k)) = \\ \hat{u}^i(t_{k-1}) &+ \text{sign}(\hat{u}^i(t_{k-1}) - u^i(t_k)) m^i(t_k) \sqrt{\eta_i(t_k)}, \end{aligned} \quad (9)$$

where  $m^i(t_k) := \left\lfloor \frac{|\hat{u}^i(t_{k-1}) - u^i(t_k)|}{\sqrt{\eta_i(t_k)}} \right\rfloor$ . The error after this update is:

$$\begin{aligned} e_u^i(t_k) &:= \hat{u}^i(t_k) - u^i(t_k) = \text{sign}(\hat{u}^i(t_{k-1}) - \\ u^i(t_k)) &\left( m^i(t_k) - \frac{|\hat{u}^i(t_{k-1}) - u^i(t_k)|}{\sqrt{\eta_i(t_k)}} \right) \sqrt{\eta_i(t_k)}. \end{aligned} \quad (10)$$

One can easily observe that,  $|e_u^i(t_k)| < \sqrt{\eta_i(t_k)}$ . That is, when there is an event locally, after the update by (9), (7) does not hold anymore. Later we show that,  $\forall i \in \{1, \dots, n_u\}$ ,  $k \in \mathbb{N}$ ,  $m^i(t_k) \leq \bar{m}_x < \infty$ . Thus, in practice one only needs to send  $\text{sign}(\hat{u}^i(t_{k-1}) - u^i(t_k))$  and  $m^i(t_k)$  for each input update. Therefore, only  $\log_2(m^i(t_k)) + 1$  bits are required for each transmission. Define  $\Gamma_{\mathcal{J}} := \text{diag}(\Gamma_{\mathcal{J}}^y, \Gamma_{\mathcal{J}}^v) = \text{diag}(\gamma_{\mathcal{J}}^1 \dots, \gamma_{\mathcal{J}}^{n_u})$ . Where  $\mathcal{J}$  is an index set:  $\mathcal{J} \subseteq \bar{\mathcal{J}} = \{1, \dots, n_u\}$  for  $u(t)$ , indicating the occurrence of events. Define  $\mathcal{J}_c := \bar{\mathcal{J}} \setminus \mathcal{J}$ . For  $l \in \{1, \dots, n_u\}$ , if  $l \in \mathcal{J}$ ,  $\gamma_{\mathcal{J}}^l = 1$ ; if  $l \in \mathcal{J}_c$ ,  $\gamma_{\mathcal{J}}^l = 0$ . Furthermore, we use the notation  $\Gamma_j = \Gamma_{\{j\}}$ . Define:

$$C := \begin{bmatrix} C_p & 0 \\ 0 & C_c \end{bmatrix}, D := \begin{bmatrix} 0 & 0 \\ D_c & 0 \end{bmatrix}.$$

The local event-triggered condition (7) can now be reformulated as a set membership:

$$\begin{aligned} i \in \mathcal{J} &\text{ iff } \xi^T(t_k) Q_i \xi(t_k) \geq \eta_i(t_k), \\ Q_i &= \begin{bmatrix} C^T \Gamma_i C & C^T \Gamma_i D - C^T \Gamma_i \\ D^T \Gamma_i C - \Gamma_i C & (D - I)^T \Gamma_i (D - I) \end{bmatrix}. \end{aligned} \quad (11)$$

Now (3)-(4) can be reformulated as:

$$\forall t_k \in \mathcal{T}, \hat{u}^i(t_k) := \begin{cases} q(u^i(t_k)), & \text{if } i \in \mathcal{J} \\ \hat{u}^i(t_{k-1}), & \text{if } i \in \mathcal{J}_c. \end{cases} \quad (12)$$

The periodic asynchronous event-triggered implementation determined by (1), (2), (5), (11), and (12) can be re-written as an impulsive system model, given by:

$$\begin{aligned} \begin{bmatrix} \dot{\xi}(t) \\ \dot{\tau} \end{bmatrix} &= \begin{bmatrix} \bar{A} \xi(t) + \bar{B} w(t) \\ 1 \end{bmatrix}, \text{ when } \tau \in [0, h[, \\ \begin{bmatrix} \xi(t_k^+) \\ \tau^+ \end{bmatrix} &= \begin{bmatrix} J_{\mathcal{J}} \xi(t_k) + \Delta_{\mathcal{J}}(t_k) \eta(t_k) \\ 0 \end{bmatrix}, \text{ when } \tau = h, \\ z(t) &= g(\xi(t), w(t)), \end{aligned} \quad (13)$$

where  $\tau$  is the elapsed time since the last sampling time, i.e.  $\tau := t - t_k$ ,  $t \in [t_k, t_{k+1}]$ , and

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A_p & 0 & 0 & B_p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} E \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\ J_{\mathcal{J}} &= \begin{bmatrix} I & 0 & 0 & 0 \\ B_c \Gamma_{\mathcal{J}}^y C_p & A_c & B_c(I - \Gamma_{\mathcal{J}}^y) & 0 \\ \Gamma_{\mathcal{J}}^y C_p & 0 & (I - \Gamma_{\mathcal{J}}^y) & 0 \\ 0 & \Gamma_{\mathcal{J}}^v C_c & \Gamma_{\mathcal{J}}^v D_c & (I - \Gamma_{\mathcal{J}}^v) \end{bmatrix}, \\ \Delta_{\mathcal{J}}(t_k) &= \begin{bmatrix} 0 \\ B_c \Gamma_{\mathcal{J}}^y \epsilon_y(t_k) \Theta_y \\ \Gamma_{\mathcal{J}}^y \epsilon_y(t_k) \Theta_y \\ \Gamma_{\mathcal{J}}^v \epsilon_v(t_k) \Theta_v \end{bmatrix}, \end{aligned}$$

with  $I$  an identity matrix of corresponding dimension and

$$\begin{aligned} \epsilon_y(t_k) &:= \text{diag} \left( \frac{e_u^1(t_k)}{\sqrt{\eta_1(t_k)}}, \dots, \frac{e_u^{n_y}(t_k)}{\sqrt{\eta_{n_y}(t_k)}} \right), \\ \epsilon_v(t_k) &:= \text{diag} \left( \frac{e_u^{n_y+1}(t_k)}{\sqrt{\eta_{n_y+1}(t_k)}}, \dots, \frac{e_u^{n_y+n_v}(t_k)}{\sqrt{\eta_{n_y+n_v}(t_k)}} \right), \\ \Theta_y &:= [\theta_1 \dots \theta_{n_y}]^T, \Theta_v := [\theta_{n_y+1} \dots \theta_{n_y+n_v}]^T. \end{aligned}$$

The term  $\Delta_{\mathcal{J}}(t_k) \eta(t_k)$  represents the quantization error after input updates and  $\frac{e_u^i(t_k)}{\sqrt{\eta_i(t_k)}} \in ]-1, 1[$  due to (9) and (10).

Lemma 9 in [19] indicates that, for a system applying the AETC mechanism to be uniformly global asymptotical stable (UGAS, see [22]) when  $w = 0$ ,  $\eta(t)$  should be a monotonically decreasing function with  $\lim_{t \rightarrow \infty} \eta(t) = 0$ . However, this mechanism does not consider system with disturbances. According to [18], when  $w \neq 0$ , if  $\eta(t)$  is arbitrarily small, the mechanism is not robust against disturbances. Meanwhile, in [19], the  $\eta(t)$  update is determined by an upper bound estimate of the current state of the plant. This estimate is not always obtainable in an output-feedback system, making it unapplicable in such systems.

We overcome the first problem by imposing a lower bound on  $\eta(t_k)$ , defined as  $\eta_{\min} > 0$ , i.e.  $\eta(t_k) \geq \eta_{\min}, \forall t_k \in \mathcal{T}$ . For the second problem, we instead use  $\xi_c(t_k)$ ,  $\hat{y}(t_k)$ , and  $\hat{v}(t_k)$  to determine the current threshold instead of  $\xi_p(t_k)$ , since this information is available to the controller.

Now we present the main problem we solve in this paper.

**Problem 5** *Design an update mechanism for  $\eta$  and a performance function  $z$  such that some compact set  $\mathcal{A}$  around the origin is UGpAS for (13) when  $w = 0$ , and the  $\mathcal{L}_2$ -gain from  $w$  to  $z$  is smaller than or equal to  $\gamma$ .*

**Remark 6** *By imposing a lower bound  $\eta_{\min}$  on  $\eta$ , the  $\lim_{t \rightarrow \infty} \eta(t) \neq 0$ , and thus  $\xi(t)$  can only converge to a set even when  $w = 0$ . Therefore, no  $\mathcal{L}_2$ -gain can be obtained for a linear performance function, proportional to the state of the system as in [11], since in that case  $\xi \notin \mathcal{L}_2$  implies  $z \notin \mathcal{L}_2$ . We circumvent this problem picking a performance function that is zero on a compact set around the origin.*

#### 4 Stability and $\mathcal{L}_2$ -gain analysis

Denote  $\tilde{z}(t)$  a reference function of  $z(t)$ , given by:

$$\tilde{z}(t) := \bar{C}\xi(t) + \bar{D}w(t). \quad (14)$$

in which,  $\bar{C}$  and  $\bar{D}$  are some matrices of appropriate dimensions. Now let us consider a Lyapunov function candidate for the impulsive system (13), (14) of the form:

$$V(x, r) = x^T P(r) x, \quad (15)$$

where  $x \in \mathbb{R}^{n_\xi}$ ,  $r \in [0, h]$ , with  $P : [0, h] \rightarrow \mathbb{R}^{n_\xi \times n_\xi}$  satisfying the Riccati differential equation:

$$\frac{d}{d\tau} P = -\bar{A}^T P - P \bar{A} - 2\rho P - \gamma^{-2} \bar{C}^T \bar{C} - G^T M G, \quad (16)$$

in which  $M := (I - \gamma^{-2} \bar{D}^T \bar{D})^{-1}$ ;  $G := \bar{B}^T P + \gamma^{-2} \bar{D}^T \bar{C}$ , with  $\bar{A}$ ,  $\bar{B}$ ,  $\bar{C}$ , and  $\bar{D}$  defined in (13) and (14), and  $\rho$

and  $\gamma$  are pre-design parameters. We often use the shorthand notation  $V(t)$  to denote  $V(\xi(t), \tau)$ . Construct the Hamiltonian matrix:

$$H := \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, F(r) := e^{-Hr} = \begin{bmatrix} F_{11}(r) & F_{12}(r) \\ F_{21}(r) & F_{22}(r) \end{bmatrix},$$

where  $H_{11} := \bar{A} + \rho I + \gamma^{-2} \bar{B} M \bar{D}^T \bar{C}$ ,  $H_{12} := \bar{B} M \bar{B}^T$ ,  $H_{21} := -\bar{C}^T (\gamma^2 I - \bar{D} \bar{D}^T)^{-1} \bar{C}$ ,  $H_{22} := -(\bar{A} + \rho I + \gamma^{-2} \bar{B} M \bar{D}^T \bar{C})^T$ .

**Assumption 7**  $F_{11}(r)$  is invertible  $\forall r \in [0, h]$ .

Since  $F_{11}(0) = I$  and  $F_{11}(r)$  is continuous, Assumption 7 can always be satisfied for sufficiently small  $h$ . Define the matrix  $\bar{S}$  satisfying  $\bar{S} \bar{S}^T := -F_{11}^{-1}(h) F_{12}(h)$ .

The following two lemmas are intermediate results from the proof of Theorem III.2 in [11].

**Lemma 8** *Consider the impulsive system (13) and (14), the Lyapunov function candidate (15) with  $P$  satisfying (16), if  $\rho > 0$ ,  $\gamma^2 > \lambda_{\max}(\bar{D}^T \bar{D})$ , then for all  $x \in \mathbb{R}^{n_\xi}$  and  $\tau \in [0, h]$ , the following inequation holds:*

$$\frac{d}{dt} V(t) \leq -2\rho V(t) - \gamma^{-2} \tilde{z}^T(t) \tilde{z}(t) + w^T(t) w(t). \quad (17)$$

**Lemma 9** *Consider the impulsive system (13) and (14), the Lyapunov function candidate (15) with  $P$  satisfying (16), and that Assumption 7 holds. If  $\gamma^2 > \lambda_{\max}(\bar{D}^T \bar{D})$  and  $\exists P(h) > 0$  satisfying  $I - \bar{S}^T P(h) \bar{S} \succ 0$ , then for  $\tau \in [0, h]$ ,  $P(\tau) \succ 0$ ; and  $P(0)$  can be expressed as:*

$$P(0) = F_{21}(h) F_{11}^{-1}(h) + F_{11}^{-T}(h) (P(h) + P(h) \bar{S} (I - \bar{S}^T P(h) \bar{S})^{-1} \bar{S}^T P(h)) F_{11}^{-1}(h). \quad (18)$$

We present next the designed threshold update mechanism. At each sampling time  $t_k^+$ , right after a jump of system (13), the controller executes the threshold update mechanism:

$$\eta(t_k^+) = \mu^{-n_\mu(t_k^+)} \eta_{\min} \\ n_\mu(t_k^+) := \max \left\{ 0, \left\lceil -\log_\mu \left( \frac{|\xi'(t_k^+)|}{\varrho \eta_{\min}} \right) - 1 \right\rceil \right\}, \quad (19)$$

in which,  $\eta_{\min}$  is a pre-designed minimum threshold; finite  $\varrho > 0$  is a design parameter; and the scalar  $\mu \in ]0, 1[$  is also a pre-designed parameter.  $\xi'(t_k^+)$  is the vector of variables available at the controller at sampling time  $t_k^+$ , denoted by:

$$\xi'(t_k^+) := \begin{bmatrix} \xi_c^T(t_k^+) & \hat{y}^T(t_k^+) & \hat{v}^T(t_k^+) \end{bmatrix}^T.$$

**Lemma 10** Consider the impulsive system (13) and (14), after the execution of the threshold update mechanism (19), if  $\eta(t_k^+) \neq \eta_{\min}$ , the following inequation holds:

$$\varrho\eta(t_k^+) < |\xi'(t_k^+)| \leq \mu^{-1}\varrho\eta(t_k^+). \quad (20)$$

Now we analyze the jump part of the impulsive system (13).

**Lemma 11** Consider the impulsive system (13) and (14) with threshold update mechanism (19). Assume all the hypotheses in Lemma 9 hold. If there exist: a matrix  $P(h) \succ 0$ , scalars  $\varrho > 0$  and  $\epsilon > 0$ , such that the linear matrix inequality (LMI):

$$\begin{bmatrix} \epsilon I & \tilde{F}_1 & \tilde{F}_2 & -\epsilon J_{\tilde{\mathcal{J}}} \\ \tilde{F}_1^T & \tilde{F}_3 & 0 & 0 \\ \tilde{F}_2^T & 0 & \tilde{F}_2 & 0 \\ -\epsilon J_{\tilde{\mathcal{J}}}^T & 0 & 0 & P(h) + \epsilon J_{\tilde{\mathcal{J}}}^T J_{\tilde{\mathcal{J}}} - \epsilon \frac{|\bar{\Delta}_{\tilde{\mathcal{J}}}|^2}{\varrho^2} I \end{bmatrix} \succeq 0 \quad (21)$$

holds, where

$$\begin{aligned} \tilde{F}_1 &:= F_{11}^{-T}(h)P(h)\bar{S} \\ \tilde{F}_2 &:= F_{11}^{-T}(h)P(h)F_{11}^{-1}(h) + F_{21}(h)F_{11}^{-1}(h) \\ \tilde{F}_3 &:= I - \bar{S}^T P(h)\bar{S} \\ \bar{\Delta}_{\tilde{\mathcal{J}}} &:= \Delta_{\mathcal{J}}(t_k)|_{\epsilon_y(t_k)=I, \epsilon_v(t_k)=I}, \end{aligned}$$

then  $\forall t_k \in \mathcal{T}$  such that  $|\xi(t_k)| > \varrho\eta(t_k)$ , the following also holds:  $V(\xi(t_k^+), 0) \leq V(\xi(t_k), h)$ .

Denote the solution set  $\mathcal{X}$  as  $(x, r) \in \mathcal{X} \subseteq \mathbb{R}^{n_\epsilon} \times [0, h]$ , such that  $x = \xi(t)$  for some  $t \in \mathbb{R}_0^+$ , where  $\xi$  is a solution to system (13) and (14). Define  $C_H = \{(x, r) | (x, r) \in \mathcal{X}, r \in [0, h]\}$ ,  $D_H = \{(x, r) | (x, r) \in \mathcal{X}, r = h\}$ . Define the set  $\mathcal{A}$  as:

$$\mathcal{A} := \{(x, r) | (x, r) \in \mathcal{X}, V(x, r) \leq \bar{\lambda}\bar{\varrho}^2\eta_{\min}^2\}, \quad (22)$$

where  $\bar{\lambda} := \max\{\lambda_{\max}(P(r)), \forall r \in [0, h]\}$ ,  $\bar{\varrho}$  is a design parameter. Inspired by [5], define a new Lyapunov function candidate for system (13), (14), and (19), as:

$$W(x, r) := \max\{V(x, r) - \bar{\lambda}\bar{\varrho}^2\eta_{\min}^2, 0\}. \quad (23)$$

Note that (23) defines a proper Lyapunov function candidate, since for this  $W(x, r)$ , all the conditions in Definition 2 are satisfied. We also use the shorthand notation  $W(t)$  to denote  $W(\xi(t), \tau)$ . Re-define the performance variable  $z(t)$  in (13) by:

$$z(t) := \begin{cases} \bar{C}\xi(t) + \bar{D}w(t), & \forall (\xi(t), r) \in \mathcal{X} \setminus \mathcal{A} \\ 0, & \forall (\xi(t), t) \in \mathcal{A}. \end{cases} \quad (24)$$

**Theorem 12** Consider system (13) and (14), threshold update mechanism (19), and performance (24). If the hypotheses of Lemma 8 and Lemma 11 hold, then  $\exists \bar{\varrho} > \varrho$ ,  $\varrho$  computed from (21), s.t.  $\mathcal{A}$  defined in (22) is a globally pre-asymptotically stable set for the impulsive system (13) when  $w = 0$ ; the  $\mathcal{L}_2$ -gain from  $w$  to  $z$  of (24) is smaller than or equal to  $\gamma$ . Moreover, an upper bound of  $\bar{\varrho}$  is given by  $\max\{|J_{\mathcal{J}}|\varrho + |\bar{\Delta}_{\mathcal{J}}|, \forall \mathcal{J} \subseteq \tilde{\mathcal{J}}\}$ .

## 5 Practical considerations

In our proposed implementation, the data a sensor sends is actually  $m^i(t_k)$  and the sign of the error, see (9). Therefore, computing an upper bound  $\bar{m}_x \geq m^i(t_k)$ ,  $\forall t_k \in \mathcal{T}$  is desirable to properly design the supporting communication protocol.

**Proposition 13** Consider system (13) and (14), threshold update mechanism (19), and performance (24). If  $w$  is bounded (i.e.  $w \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ ), and the hypotheses of Theorem 12 hold, then:

$$\begin{aligned} \bar{m}_x &= \max\{\bar{m}_x^i | i \in \{1, \dots, n_u\}\} \\ \bar{m}_x^i &= \frac{(1 + \|C D\|)}{\theta_i} \sqrt{\frac{W(0)}{\eta_{\min}^2 \bar{\Delta}} + \frac{\|w\|_{\mathcal{L}_\infty}^2}{2\rho\eta_{\min}^2 \bar{\Delta}} + \frac{\bar{\lambda}\bar{\varrho}^2}{\bar{\Delta}}} \quad (25) \\ &\geq m^i(t_k), \forall t_k \in \mathcal{T}, \end{aligned}$$

where  $\bar{\Delta} = \min\{\lambda_{\min}(P(r)), \forall r \in [0, h]\}$ .

Similarly, an upper bound of  $n_\mu(t)$ , denoted by  $\bar{m}_\mu$  can be obtained:

**Proposition 14** Consider system (13) and (14), threshold update mechanism (19), and performance (24). If  $w$  is bounded (i.e.  $w \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ ), and the hypotheses of Theorem 12 hold, then  $\bar{m}_\mu$  is given as:

$$\begin{aligned} \bar{m}_\mu &= \max\left\{0, \right. \\ &\quad \left. -\log_\mu \left( \frac{(1 + \|C D\|)}{\varrho} \sqrt{\frac{W(0)}{\eta_{\min}^2 \bar{\Delta}} + \frac{\|w\|_{\mathcal{L}_\infty}^2}{2\rho\eta_{\min}^2 \bar{\Delta}} + \frac{\bar{\lambda}\bar{\varrho}^2}{\bar{\Delta}}} \right) \right\}. \quad (26) \end{aligned}$$

**Remark 15** For a packet from node  $i$ , the packet length is computed by  $\lceil \log_2 m^i(t_k) \rceil + 1$ , with the additional bit used to indicate the sign of the error.

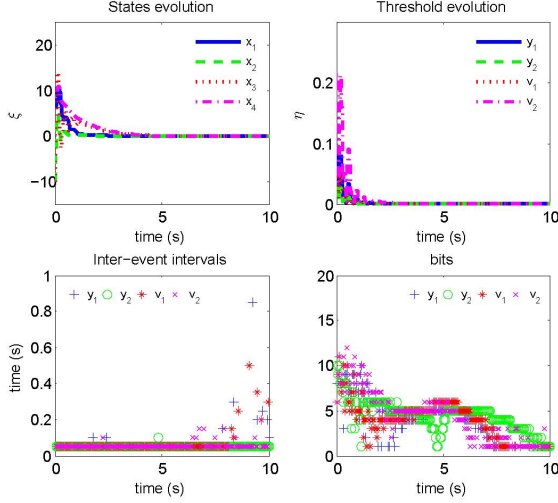


Figure 2. Simulation result when  $w = 0$ : system state, threshold, inter-event intervals, and bits of each event.

## 6 Numerical example

In this section, we consider the batch reactor system from [24]. Given  $h = 0.05s$ , the controller is obtained as:

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0.05 \\ 0 & 1 & 0.05 & 0 \\ -2 & 0 & 0 & -2 \\ 0 & 8 & 5 & 0 \end{bmatrix}.$$

With  $\rho = 0.01$ ,  $\gamma = 0.9$ ,  $z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi$ , Assumption 7 is satisfied. Solving (21), one can obtain a  $\varrho = 200.2$ . Other parameters are given by  $\mu = 0.75$ ,  $\eta_{\min} = 0.0001$ ,  $\theta_1 = 0.3397$ ,  $\theta_2 = 0.1132$ ,  $\theta_3 = 0.2265$ , and  $\theta_4 = 0.9058$ .  $\xi(0) = \begin{bmatrix} 10 & -10 & -10 & 10 & 0 & 0 & -10 & -10 & 20 & -50 \end{bmatrix}^T$ . Resulting in the set  $\mathcal{A} = \{(x, r) | (x, r) \in \mathcal{X}, |x^T P(r)x| \leq 3.1135\}$ .

Fig 2 shows the simulation results of the system without disturbances. The system state converges to the set  $\mathcal{A}$ , however one can see that the estimate of  $\mathcal{A}$  is very conservative compared with the simulation result ( $|x^T P(r)x| = 2.36 \times 10^{-6}$  at 10s). The sensor transmissions are reduced by 11.07% compared to a time-triggered mechanism with the same sampling interval  $h$ . The maximum inter-event interval is 0.85 seconds. The following bounds are obtained from our analysis:  $\bar{m}_x = 2.3997 \times 10^8$  (29 bits), and  $\bar{m}_\mu = 42$ . The maximum observed  $m^i(t_k)$  is 1303 (12 bits), i.e.  $1.8417 \times 10^5$  times smaller than the computed  $\bar{m}_x$ . Most of the time (95.38%),  $m^i(t_k)$  is smaller than or equal to 128 (8 bits); 46.15% of the  $m^i(t_k)$  can be transmitted with solely 4 bits.

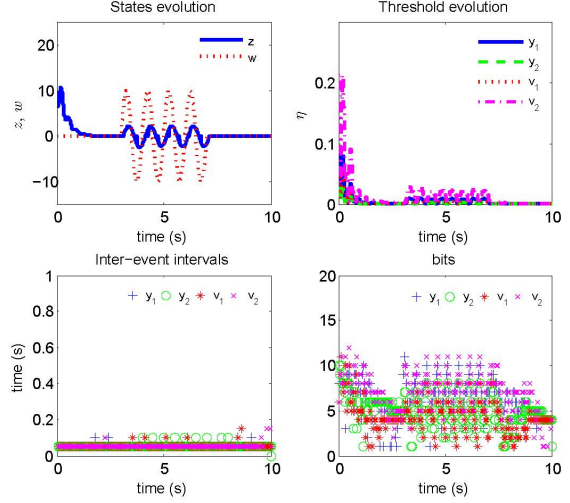


Figure 3. Simulation result when  $w(t) = 10 \sin(2\pi t)$ ,  $t \in [3, 7]$ : evolution of  $z$  and  $w$ , threshold, inter-event intervals, and bits of each event.

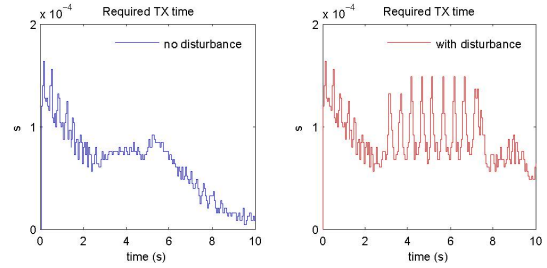


Figure 4. Required transmitting time with rate 250 kbit/s, both without and with disturbance.

Fig 3 shows the simulation results in the presence of a finite sine wave disturbance. It can be seen that the performance variable  $z$  follows  $w$  with a bounded norm ratio. The sensor transmissions are reduced by 3.61% compared to a time-triggered mechanism with the same sampling interval  $h$ . The maximum inter-event interval is 0.15 seconds. 89.81% of  $m^i(t_k)$  are smaller than or equal to 128 (8 bits); 31.23% of  $m^i(t_k)$  can be transmitted with 4 bits; and the maximum  $m^i(t_k)$  is still 1303 (12 bits). Note that the saving of transmission increases as the time without disturbances increases. Further simulation results show that, the sensor transmissions are reduced by 63.81% after running for 50s without additional disturbances.

Fig 4 shows the simulation results of the total transmitting time required during each sampling time  $h$ , under ZigBee (over-the-air data rate 250 kbit/s). One can observe that, the required transmission time reduces as the state converges to  $\mathcal{A}$ . The maximum required transmission time is  $1.64 \times 10^{-4}s$ , about 304.88 times smaller than the sampling time  $h$ .

## 7 Conclusion and future work

We propose periodic asynchronous event-triggered control implementations as an extension to the work of [11] and [19]. This triggering strategy combines decentralized event generation, asynchronous sampling update, and zoom in/out quantization. This approach lets the implementation exchange very few bits every time that an event triggers a transmission, reduces the required amount of transmission compared with time-triggered mechanisms, and reduces the necessary sensing compared with continuously monitored event-triggered mechanisms. The maximum amounts of bits that may be needed to update samplings and thresholds after an event is triggered are provided. Such a bound enables the design of actual implementations for wireless systems, whose demonstration on physical experiments is part of our future work. How to reduce the conservatism of the estimate of the attractive set  $\mathcal{A}$ , how to optimize  $\mu$ , and how to compensate transmission delays are additional goals of future work.

## Appendix. Proofs

**Proof of Lemma 10** For any  $s = \left\lceil -\log_\mu \left( \frac{|\xi'(t_k^+)|}{\varrho\eta_{\min}} \right) - 1 \right\rceil$ ,  $s$  satisfies:

$$-\log_\mu \left( \frac{|\xi'(t_k^+)|}{\varrho\eta_{\min}} \right) - 1 \leq s < -\log_\mu \left( \frac{|\xi'(t_k^+)|}{\varrho\eta_{\min}} \right),$$

with which, noting that  $\mu \in ]0, 1[$ , it is easy to obtain that:

$$\mu^{\log_\mu \left( \frac{|\xi'(t_k^+)|}{\varrho\eta_{\min}} \right) + 1} \leq \mu^{-s} < \mu^{\log_\mu \left( \frac{|\xi'(t_k^+)|}{\varrho\eta_{\min}} \right)},$$

which, as  $\varrho\eta_{\min} > 0$ , can be finally simplified as:

$$\mu|\xi'(t_k^+)| \leq \varrho\mu^{-s}\eta_{\min} < |\xi'(t_k^+)|. \quad (27)$$

From (19), after the execution of the threshold update mechanism:

$$\eta(t_k^+) = \max\{\eta_{\min}, \mu^{-s}\eta_{\min}\}.$$

If  $\eta(t_k^+) \neq \eta_{\min}$ , then  $\eta(t_k^+) = \mu^{-s}\eta_{\min}$ , and thus from (27):

$$\mu|\xi'(t_k^+)| \leq \varrho\eta(t_k^+) < |\xi'(t_k^+)|,$$

which can be re-written as (20). This ends the proof.  $\square$

**Proof of Lemma 11** For the jump part of the impulse system (13), we have that the relation between the states

before and after each jump is given by:

$$\begin{aligned} |\xi(t_k^+) - J_{\bar{\mathcal{J}}}\xi(t_k)| &= |J_{\mathcal{J}}\xi(t_k) + \Delta_{\mathcal{J}}(t_k)\eta(t_k) - J_{\bar{\mathcal{J}}}\xi(t_k)| \\ &= |\tilde{H}_1\xi(t_k) + \Delta_{\mathcal{J}}(t_k)\eta(t_k)|, \end{aligned}$$

where

$$\tilde{H}_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ -B_c\Gamma_{\mathcal{J}_c}^y C_p & 0 & B_c\Gamma_{\mathcal{J}_c}^y & 0 \\ -\Gamma_{\mathcal{J}_c}^y C_p & 0 & \Gamma_{\mathcal{J}_c}^y & 0 \\ 0 & -\Gamma_{\mathcal{J}_c}^v C_c & -\Gamma_{\mathcal{J}_c}^v D_c & \Gamma_{\mathcal{J}_c}^v \end{bmatrix},$$

since  $\Gamma_{\mathcal{J}_c}^y + \Gamma_{\bar{\mathcal{J}}}^y = I = \Gamma_{\bar{\mathcal{J}}}^y$  and  $\Gamma_{\mathcal{J}_c}^v + \Gamma_{\bar{\mathcal{J}}}^v = I = \Gamma_{\bar{\mathcal{J}}}^v$ . By the definition of error (10) and the event-triggered mechanism (11), one has:

$$\begin{aligned} \Gamma_{\mathcal{J}_c}^y \hat{y}(t_k) - \Gamma_{\mathcal{J}_c}^y y(t_k) &= \Gamma_{\mathcal{J}_c}^y \epsilon_y(t_k) \Theta_y \eta(t_k) \\ \Gamma_{\mathcal{J}_c}^v \hat{v}(t_k) - \Gamma_{\mathcal{J}_c}^v v(t_k) &= \Gamma_{\mathcal{J}_c}^v \epsilon_v(t_k) \Theta_v \eta(t_k), \end{aligned}$$

therefore:

$$\begin{aligned} \tilde{H}_1\xi(t_k) + \Delta_{\mathcal{J}}(t_k)\eta(t_k) &= \Delta_{\mathcal{J}_c}(t_k)\eta(t_k) + \Delta_{\bar{\mathcal{J}}}(t_k)\eta(t_k) \\ &= \Delta_{\bar{\mathcal{J}}}(t_k)\eta(t_k). \end{aligned}$$

Thus:

$$|\xi(t_k^+) - J_{\bar{\mathcal{J}}}\xi(t_k)| = |\Delta_{\bar{\mathcal{J}}}(t_k)\eta(t_k)| \leq |\bar{\Delta}_{\bar{\mathcal{J}}}|\eta(t_k).$$

Together with the hypothesis that  $|\xi(t_k)| > \varrho\eta(t_k)$ , one has:

$$|(\xi(t_k^+) - J_{\bar{\mathcal{J}}}\xi(t_k))|^2 < \frac{|\bar{\Delta}_{\bar{\mathcal{J}}}|^2}{\varrho^2} |\xi(t_k)|^2. \quad (28)$$

Which can be re-written as:

$$\begin{bmatrix} \xi(t_k^+) \\ \xi(t_k) \end{bmatrix}^T \begin{bmatrix} I & -J_{\bar{\mathcal{J}}} \\ -J_{\bar{\mathcal{J}}}^T & J_{\bar{\mathcal{J}}}^T J_{\bar{\mathcal{J}}} - \frac{\bar{\Delta}_{\bar{\mathcal{J}}}^2}{\varrho^2} I \end{bmatrix} \begin{bmatrix} \xi(t_k^+) \\ \xi(t_k) \end{bmatrix} < 0. \quad (29)$$

From the hypotheses, particularly (21) together with the result from Lemma 9, and Schur complement, we have that (21) implies:

$$\epsilon \begin{bmatrix} I & -J_{\bar{\mathcal{J}}} \\ -J_{\bar{\mathcal{J}}}^T & J_{\bar{\mathcal{J}}}^T J_{\bar{\mathcal{J}}} - \frac{\bar{\Delta}_{\bar{\mathcal{J}}}^2}{\varrho^2} I \end{bmatrix} + \begin{bmatrix} -P(0) & 0 \\ 0 & P(h) \end{bmatrix} \succeq 0. \quad (30)$$

Since  $\epsilon > 0$ , by applying the S-procedure (see e.g. [2]) from (29) and (30), one can conclude that:

$$\begin{bmatrix} \xi(t_k^+) \\ \xi(t_k) \end{bmatrix}^T \begin{bmatrix} -P(0) & 0 \\ 0 & P(h) \end{bmatrix} \begin{bmatrix} \xi(t_k^+) \\ \xi(t_k) \end{bmatrix} \geq 0. \quad (31)$$

This ends the proof.  $\square$



**Proof of Theorem 12** We first show that  $\mathcal{A}$  is a uniformly global pre-asymptotically stable set for the impulsive system (13) when  $w = 0$ . A new Lyapunov function candidate  $W$ , given by (23), is introduced. Define  $\mathcal{B} := \{(x, r) | (x, r) \in \mathcal{X}, |x| \leq \varrho\eta_{\min}\}$ . If  $\eta(t_k) = \eta_{\min}$ ,  $|\xi(t_k)| > \varrho\eta_{\min}$  implies  $|\xi(t_k)| > \varrho\eta(t_k)$ ; if  $\eta(t_k) > \eta_{\min}$ , according to Lemma 10,  $\varrho\eta(t_k) < |\xi'(t_k)| \leq |\xi(t_k)|$ . Therefore,  $\forall (\xi(t_k), \tau) \in D_H \setminus \mathcal{B}$ ,  $|\xi(t_k)| > \varrho\eta(t_k)$ , and thus from Lemma 11,  $\forall (\xi(t_k), \tau) \in D_H \setminus \mathcal{B}$ :

$$V(\xi(t_k^+), 0) \leq V(\xi(t_k), h). \quad (32)$$

According to Lemma 10, if  $|\xi'(t_k)| \leq \varrho\eta(t_k)$  then  $\eta(t_k) = \eta_{\min}$ , i.e.  $\forall (\xi(t_k), \tau) \in D_H \cap \mathcal{B}$ ,  $\eta(t_k) = \eta_{\min}$ . Furthermore,  $(\xi(t_k), \tau) \in D_H \cap \mathcal{B}$  implies  $\xi(t_k^+) = J_{\mathcal{J}}\xi(t_k) + \Delta_{\mathcal{J}}\eta_{\min}$ , and thus,  $|\xi(t_k^+)| \leq |J_{\mathcal{J}}|\xi(t_k)| + |\Delta_{\mathcal{J}}|\eta_{\min} \leq (|J_{\mathcal{J}}|\varrho + |\Delta_{\mathcal{J}}|)\eta_{\min} \leq \bar{\varrho}\eta_{\min}$ . That is,  $\forall (\xi(t_k), \tau) \in D_H \cap \mathcal{B}$ ,  $(\xi(t_k^+), 0) \in \mathcal{A}$ . Note that, since  $|J_{\mathcal{J}}| > 1$ ,  $\forall (x, r) \in \mathcal{B}$ ,  $x^T P(r)x \leq \bar{\lambda}|x|^2 \leq \bar{\lambda}\varrho^2\eta_{\min}^2 < \bar{\lambda}\bar{\varrho}^2\eta_{\min}^2$ , i.e.  $\mathcal{B} \subset \mathcal{A}$ . Thus one can conclude that  $\forall (\xi(t), \tau) \in \mathcal{A} \cap D_H$ ,  $(\xi(t^+), 0) \in \mathcal{A}$ . If all the hypotheses in Lemma 8 hold, together with (23), one has  $\forall (\xi(t), \tau) \in C_H \setminus \mathcal{A}$ :

$$\begin{aligned} \frac{d}{dt}W(\xi(t), \tau) &= \frac{d}{dt}V(\xi(t), \tau) \leq -2\rho V(\xi(t), \tau) \\ &- \gamma^{-2}\tilde{z}^T(t)\tilde{z}(t) + w^T(t)w(t) < -2\rho W(\xi(t), \tau) \\ &- \gamma^{-2}\tilde{z}^T(t)\tilde{z}(t) + w^T(t)w(t). \end{aligned} \quad (33)$$

By (23) and (32), one has  $\forall (\xi(t_k), \tau) \in D_H \setminus \mathcal{A}$ :

$$\begin{aligned} W(\xi(t_k^+), 0) &= \max\{V(\xi(t_k^+), 0) - \bar{\lambda}\bar{\varrho}^2\eta_{\min}^2, 0\} \\ &\leq V(\xi(t_k), h) - \bar{\lambda}\bar{\varrho}^2\eta_{\min}^2 = W(\xi(t_k), h). \end{aligned} \quad (34)$$

Combine (33), (34) to see that, when  $w = 0$  all the conditions in Theorem 3 are satisfied. Thus,  $\mathcal{A}$  is a uniformly global pre-asymptotically stable set for the impulsive system (13).

Now we compute the  $\mathcal{L}_2$ -gain. A new performance variable  $z$ , given by (24), is introduced. Define a set of times:

$$\mathcal{T}_s = \{(t_i^s, j_i^s) | i \in \mathbb{N}\}, \quad (35)$$

where  $(t_0^s, j_0^s)$  is the initial time, s.t.  $\forall t \in [t_{2i+1}^s, t_{2i+2}^s]$ ,  $i \in \mathbb{N}$ ,  $(\xi(t), \tau) \in \mathcal{A}$ , and the rest of the time  $(\xi(t), \tau) \in \mathcal{X} \setminus \mathcal{A}$ . If  $|\mathcal{T}_s|$  is infinite, i.e.  $(\xi(t), \tau)$  visits  $\mathcal{A}$  infinitely often, one has:

$$\begin{aligned} \int_0^\infty z^T(t)z(t)dt &= \sum_{i=0}^\infty \int_{t_i^s}^{t_{i+1}^s} z^T(t)z(t)dt = \\ &\sum_{i=0}^\infty \int_{t_{2i}^s}^{t_{2i+1}^s} z^T(t)z(t)dt + \sum_{i=0}^\infty \int_{t_{2i+1}^s}^{t_{2i+2}^s} z^T(t)z(t)dt. \end{aligned} \quad (36)$$

By (33),  $\forall (\xi(t), \tau) \in C_H \setminus \mathcal{A}$ :

$$\frac{d}{dt}W(\xi(t), \tau) < -\gamma^{-2}z^T(t)z(t) + w^T(t)w(t). \quad (37)$$

One can replace the integration of  $\frac{d}{dt}W(t)$ ,  $z^T(t)z(t)$ , and  $w^T(t)w(t)$  on the open interval  $]t_{2i}^s, t_{2i+1}^s[$  by the integration on the closure of that interval, see [1]. Applying the Comparison Lemma (e.g. Lemma 2.5 in [14]) to (34) and (37), one has:

$$\begin{aligned} W(t_{2i+1}^s) - W(t_{2i}^s) &= \int_{t_{2i}^s}^{t_{2i+1}^s} \frac{d}{dt}W(t)dt \\ &< \int_{t_{2i}^s}^{t_{2i+1}^s} (-\gamma^{-2}z^T(t)z(t) + w^T(t)w(t))dt. \end{aligned} \quad (38)$$

Since  $\forall i \in \mathbb{N}, i \neq 0$ ,  $W(t_i^s) = 0$ , therefore  $\forall i \in \mathbb{N}$ :

$$\begin{aligned} \sum_{i=0}^\infty \int_{t_{2i}^s}^{t_{2i+1}^s} z^T(t)z(t)dt &< \gamma^2 \sum_{i=0}^\infty \int_{t_{2i}^s}^{t_{2i+1}^s} w^T(t)w(t)dt \\ &+ \gamma^2 W(t_0^s). \end{aligned} \quad (39)$$

When  $(\xi(t), \tau) \in \mathcal{A}$ , we have  $z(t) = 0$  from (24), thus:

$$\sum_{i=0}^\infty \int_{t_{2i+1}^s}^{t_{2i+2}^s} z^T(t)z(t)dt = 0 \leq \gamma^2 \sum_{i=0}^\infty \int_{t_{2i+1}^s}^{t_{2i+2}^s} w^T(t)w(t)dt. \quad (40)$$

Put (39) and (40) into (36) to obtain:

$$\|z\|_{\mathcal{L}_2}^2 < \gamma^2 W(t_0^s) + \gamma^2 \|w\|_{\mathcal{L}_2}^2 \leq (\delta(\xi(0)) + \gamma \|w\|_{\mathcal{L}_2})^2. \quad (41)$$

If  $\exists T$  s.t.  $\forall t > T$ ,  $(\xi(t), \tau) \in \mathcal{X} \setminus \mathcal{A}$ , then  $|\mathcal{T}_s| = 2I_s$  for some finite  $I_s \in \mathbb{N}$ . Since  $\forall t \in \mathbb{R}_0^+$ ,  $W(t) \geq 0$ , and  $W(t_{2I_s}^s) = 0$ :  $-\int_{t_{2I_s}^s}^\infty \frac{d}{dt}W(t)dt \leq 0$ , and thus  $\int_{t_{2I_s}^s}^\infty z^T(t)z(t)dt \leq \gamma^2 \int_{t_{2I_s}^s}^\infty w^T(t)w(t)dt$ . Therefore:

$$\begin{aligned} \|z\|_{\mathcal{L}_2}^2 &= \left( \sum_{i=0}^{I_s-1} \int_{t_{2i}^s}^{t_{2i+1}^s} z^T(t)z(t)dt + \int_{t_{2I_s}^s}^\infty z^T(t)z(t)dt \right) \\ &+ \sum_{i=0}^{I_s-1} \int_{t_{2i+1}^s}^{t_{2i+2}^s} z^T(t)z(t)dt < (\delta(\xi(0)) + \gamma \|w\|_{\mathcal{L}_2})^2. \end{aligned}$$

If  $\exists T$  s.t.  $\forall t > T$ ,  $(\xi(t), \tau) \in \mathcal{A}$ , then  $|\mathcal{T}_s| = 2I_s + 1$  for some finite  $I_s \in \mathbb{N}$ , and thus  $\int_{t_{2I_s+1}^s}^\infty z^T(t)z(t)dt = 0$ . Therefore:

$$\begin{aligned} \|z\|_{\mathcal{L}_2}^2 &= \left( \sum_{i=0}^{I_s-1} \int_{t_{2i}^s}^{t_{2i+2}^s} z^T(t)z(t)dt + \int_{t_{2I_s+1}^s}^\infty z^T(t)z(t)dt \right) \\ &+ \sum_{i=0}^{I_s} \int_{t_{2i}^s}^{t_{2i+1}^s} z^T(t)z(t)dt < (\delta(\xi(0)) + \gamma \|w\|_{\mathcal{L}_2})^2. \end{aligned}$$

Which ends the proof.  $\square$

**Proof of Proposition 13** Following the proof of Theorem 12, by (33) one has  $\forall(\xi(t), \tau) \in C_H \setminus \mathcal{A}$ :

$$\frac{d}{dt}W(\xi(t), \tau) < -2\rho W(\xi(t), \tau) + w^T(t)w(t). \quad (42)$$

Using the Comparison Lemma to (34) and (42) on the interval  $[t_{2i}^s, T]$ , where  $T \in [t_{2i}^s, t_{2i+1}^s]$  to obtain:

$$\begin{aligned} W(T) &< e^{-2\rho(T-t_{2i}^s)}W(t_{2i}^s) + \frac{\|w\|_{\mathcal{L}_\infty}^2}{2\rho} \left(1 - e^{-2\rho(T-t_{2i}^s)}\right) \\ &\leq W(t_{2i}^s) + \frac{\|w\|_{\mathcal{L}_\infty}^2}{2\rho} \leq W(t_0^s) + \frac{\|w\|_{\mathcal{L}_\infty}^2}{2\rho}, \end{aligned} \quad (43)$$

since  $T \geq t_{2i}^s$  indicates  $e^{-2\rho(T-t_{2i}^s)} \in [0, 1]$ . When  $(\xi(t), \tau) \in \mathcal{A}$ ,  $W(t)$  is bounded by  $W(t) = 0 \leq \frac{\|w\|_{\mathcal{L}_\infty}^2}{2\rho}$ . Thus:

$$W(t) \leq W(0) + \frac{1}{2\rho}\|w\|_{\mathcal{L}_\infty}^2, \quad \forall(\xi(t), \tau) \in \mathcal{X}. \quad (44)$$

From the definition of  $W(x, r)$  in (23):

$$\max\{V(t) - \bar{\lambda}\bar{\rho}^2\eta_{\min}^2, 0\} = W(t) \leq W(0) + \frac{1}{2\rho}\|w\|_{\mathcal{L}_\infty}^2,$$

together with the fact that  $V(t) \geq \lambda|\xi(t)|^2$ , one obtains:

$$\forall t \in \mathbb{R}_0^+, |\xi(t)|^2 \leq \frac{W(0) + \frac{1}{2\rho}\|w\|_{\mathcal{L}_\infty}^2 + \bar{\lambda}\bar{\rho}^2\eta_{\min}^2}{\lambda}. \quad (45)$$

According to the definition of  $m^i(t_k)$ :

$$\begin{aligned} m^i(t_k) &= \left\lfloor \frac{|\hat{u}^i(t_{k-1}) - u^i(t_k)|}{\sqrt{\eta_i(t_k)}} \right\rfloor \\ &\leq \frac{|\hat{u}^i(t_{k-1})| + |u^i(t_k)|}{\sqrt{\eta_i(t_k)}} \leq \frac{|\xi(t_{k-1})| + |[CD]||\xi(t_k)|}{\sqrt{\eta_i(t_k)}}. \end{aligned} \quad (46)$$

By introducing (45) into (46), one can conclude the bound of  $m^i(t_k)$  as (25). Which ends the proof.  $\square$

**Proof of Proposition 14** By the definition of  $n_\mu(t_k)$  in (19), one has  $n_\mu(t_k) \leq \max\left\{0, -\log_\mu\left(\frac{|\xi'(t_k)|}{\varrho\eta_{\min}}\right)\right\} \leq \max\left\{0, -\log_\mu\left(\frac{|\xi(t)|}{\varrho\eta_{\min}}\right)\right\}$ . Following along the same lines of the proof of Proposition 13, the bound of  $n_\mu(t_k)$  is obtained as (26).  $\square$

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